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Analytic Solutions of the Klein-Gordon Equation in a Semi-infinite Channel

by

B. Neta, J. M. Lindquist, and F. X. Giraldo

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This report was prepared by:

B. Neta Professor	J. M. Lindquist Major, USA	F. X. Giraldo Associate Professor

Reviewed by:

Carlos Borges, Chair
Department of Applied Mathematics

Released by:

Karl A. van Bibber
Vice President and
Dean of Research

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Abstract

In this report we show how to construct analytic solutions of the Klein-Gordon equation in a semi-infinite channel. The Klein-Gordon equation can be derived from the shallow water equations. The analytic solutions are given for various choices of initial and boundary conditions.

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1 Statement of the Problem

Consider the shallow water equations (SWEs) in a semi-infinite channel. For simplicity we assume that the channel has a flat bottom and that there is no advection, although these assumptions may be removed in future studies. We do take into account rotation (Coriolis) effects. A Cartesian coordinate system (x, y) is introduced such that the channel is parallel to the x direction, as shown in the figure. The width of the channel is denoted b.

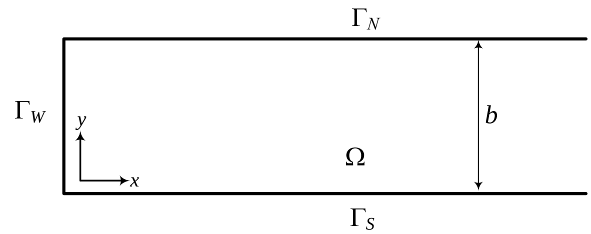

Figure 1: Setup for the wave-guide problem in a semi-infinite wave guide

The SWEs are (see [1]):

$$\begin{aligned}
&\partial_t u + \mu u \partial_x u + \mu v \partial_y u - f v = -g\ \partial_x \eta\ , \\
&\partial_t v + \mu u \partial_x v + \mu v \partial_y v + f u = -g\ \partial_y \eta\ , \\
&\partial_t \eta + \mu u \partial_x \eta + \mu v \partial_y \eta + (h_0 + \mu\eta)\ (\partial_x u + \partial_y v) = 0\ .
\end{aligned} \tag{1}$$

Here t is time, $u(x, y, t)$ and $v(x, y, t)$ are the unknown velocities in the x and y directions, h_0 is the given water layer thickness (in the direction normal to the xy plane), $\eta(x, y, t)$ is the unknown water elevation above h_0, f is the Coriolis parameter, and g is the gravity acceleration. We use the following shorthand for partial derivatives

$$\partial_a^i = \frac{\partial^i}{\partial a^i}$$

The parameter μ is 1 for the nonlinear SWEs, and is 0 for the linearized SWEs with vanishing mean flow. We shall only consider the latter in the sequel.

It can be shown (see [2]) that *a single* boundary condition must be imposed along the entire boundary to obtain a well-posed problem. On the south and north channel walls Γ_S and Γ_N we have $v = 0$ (no normal flow). On the west boundary Γ_W we prescribe η using the Dirichlet condition $\eta(0, y, t) = \eta_W(y, t)$, where $\eta_W(y, t)$ is a given function (incoming wave). At $x \to \infty$ the solution is known to be bounded and not to include any incoming waves. To complete the statement of the problem, initial values for u, v and η are given at time $t = 0$ in the entire domain.

It is easy to see (e.g., [3]) that the system (1) is equivalent to

$$\partial_t^2 \eta - C_0^2 \nabla^2 \eta + f^2 \eta = 0 \tag{2}$$

where $C_0 = \sqrt{gh_0}$. The boundary conditions are

$$\begin{aligned}
&\eta(0, y, t) = F(y, t), \\
&\partial_n \eta(x, 0, t) = 0, \\
&\partial_n \eta(x, b, t) = 0, \\
&\lim_{x \to \infty} \eta(x, y, t) \text{ is bounded}
\end{aligned} \tag{3}$$

and the initial conditions are

$$\begin{aligned}
&\eta(x, y, 0) = G(x, y), \\
&\partial_t \eta(x, y, 0) = 0
\end{aligned} \tag{4}$$

The general solution to the problem can be found by taking the Fourier sine transform in x and then solve the resulting PDE in t and y. This is very messy and will not be pursued here. In the following sections we will discuss the possible choices for $F(y, t)$ and $G(x, y)$. One can also take a non vanishing second initial condition.

2 Construction of Solutions

Since the problem (2)-(4) has a non-homogeneous boundary condition, we will decompose η

$$\eta = v + w \tag{5}$$

where v will satisfy the PDE with a homogeneous boundary condition on the west boundary and w satisfies the non-homogeneous boundary condition with a simplified PDE. We will choose w to satisfy the following problem

$$\begin{aligned}
&\partial_t^2 w - C_0^2 \partial_y^2 w + f^2 w = 0 \\
&w(0, y, t) = F(y, t), \\
&\partial_n w(x, 0, t) = 0, \\
&\partial_n w(x, b, t) = 0, \\
&\lim_{x \to \infty} w(x, y, t) \text{ is bounded} \\
&w(x, y, 0) = H(y), \\
&\partial_t w(x, y, 0) = 0
\end{aligned} \tag{6}$$

The simplification is the fact that w is independent of x.

Remark: If the boundary condition at $x = 0$ is a function of y only, then the above equation (6) will not work. We will consider that case in the next section.

Problem (6) can be solved using separation of variables, i.e. by assuming that $w = Y(y)T(t)$.

It is easy to see that the solution is

$$w = \sum_{m=0}^{\infty} A_m \cos(\sqrt{\nu_0(m)}\, t) \cos\left(\frac{m\pi}{b} y\right) \tag{7}$$

where

$$\nu_0(m) = f^2 + \left(\frac{m\pi C_0}{b}\right)^2 \tag{8}$$

Since w is independent of x, this is also $F(y,t)$. Using the initial condition

$$H(y) = \sum_{m=0}^{\infty} A_m \cos\left(\frac{m\pi}{b} y\right) \tag{9}$$

we can find the Fourier coefficients A_m. Let's choose (for simplicity)

$$H(y) = \cos\left(\frac{\pi}{b} y\right). \tag{10}$$

This choice will simplify the computation of the Fourier coefficients A_m, i.e.

$$A_m = \begin{cases} 1 & m = 1 \\ 0 & m \neq 1 \end{cases} \tag{11}$$

Therefore

$$F(y,t) = \cos(\sqrt{\nu_0(1)}\, t)\, \cos\left(\frac{\pi}{b} y\right) \tag{12}$$

Clearly if we choose a different $H(y)$, we get a different function $F(y,t)$. The solution w is then given by

$$w = \cos(\sqrt{\nu_0(1)}\, t)\, \cos\left(\frac{\pi}{b} y\right) \tag{13}$$

Now we take v given by (5) and substitute in (2), we have

$$\partial_t^2 v + \partial_t^2 w - C_0^2 \nabla^2 v - C_0^2 \nabla^2 w + f^2 v + f^2 w = 0$$

Move all the terms with w to the right and use (13), we have

$$\partial_t^2 v - C_0^2 \nabla^2 v + f^2 v = 0 \tag{14}$$

which is identical to (2). Now the boundary conditions become

$$\begin{aligned} & v(0,y,t) = \underbrace{\eta(0,y,t)}_{F(y,t)} - \underbrace{w(0,y,t)}_{F(y,t)} = 0 \\ & \partial_n v(x,0,t) = 0 \\ & \partial_n v(b,y,t) = 0 \\ & \lim_{x\to\infty} v(x,y,t) \text{ is bounded} \end{aligned} \tag{15}$$

The initial conditions are

$$\begin{aligned} & v(x,y,0) = G(x,y) - \underbrace{w(x,y,0)}_{\cos(\frac{\pi}{b} y)} \\ & \partial_t v(x,y,0) = 0 \end{aligned} \tag{16}$$

To solve (14)-(16), we use separation of variables

$$v(x,y,t) = T(t)\phi(x,y) \tag{17}$$

Substituting in (14), we have two differential equations

$$\ddot{T} + \nu T = 0 \tag{18}$$

with

$$\dot{T}(0) = 0 \tag{19}$$

and

$$\nabla^2 \phi + \frac{\nu - f^2}{C_0^2}\phi = 0 \tag{20}$$

with the boundary conditions

$$\begin{aligned}
&\phi(0, y) = 0 \\
&\partial_y \phi(x, 0) = 0 \\
&\partial_y \phi(x, b) = 0 \\
&\lim_{x\to\infty} \phi(x, y) \text{ is bounded}
\end{aligned} \tag{21}$$

To solve (20), we will separate the variables again, assuming $\phi(x, y) = X(x)Y(y)$ to get

$$\begin{array}{ll}
Y'' + \mu Y = 0 & X'' + \left(\dfrac{\nu - f^2}{C_0^2} - \mu\right) X = 0 \\
Y'(0) = 0 & X(0) = 0 \\
Y'(b) = 0 & \lim_{x\to\infty} X(x) \text{ is bounded}
\end{array} \tag{22}$$

The solution for the Y equation is

$$\begin{aligned}
&\mu_m = \left(\frac{m\pi}{b}\right)^2 \\
&Y_m(y) = \cos\left(\frac{m\pi}{b} y\right) \\
&m = 0, 1, 2, \ldots
\end{aligned} \tag{23}$$

In order for the X equation to have a non trivial solution, we must have $\nu > \nu_0(m)$ where $\nu_0(m)$ is given by (8). In this case the solution will be

$$X(x) = \sin\left(\frac{\sqrt{\nu - \nu_0(m)}}{C_0} x\right) \tag{24}$$

Therefore the solution for (20) is

$$\phi(x, y) = \sum_{m=0}^{\infty} \left[\int_{\nu_0(m)}^{\infty} A_m(\nu) \sin\left(\frac{\sqrt{\nu - \nu_0(m)}}{C_0} x\right) d\nu\right] \cos\left(\frac{m\pi}{b} y\right) \tag{25}$$

The solution of the T equation is

$$T(t) = \cos\left(\sqrt{\nu}\, t\right) \tag{26}$$

and therefore

$$v(x, y, t) = \sum_{m=0}^{\infty} \left[\int_{\nu_0(m)}^{\infty} A_m(\nu) \cos\left(\sqrt{\nu}\, t\right) \sin\left(\frac{\sqrt{\nu - \nu_0(m)}}{C_0} x\right) d\nu\right] \cos\left(\frac{m\pi}{b} y\right) \tag{27}$$

The only condition left to satisfy is $v(x,y,0) = G(x,y) - \cos\left(\frac{\pi}{b}y\right)$. Before we do that, let us transform our general solution, by taking

$$\Lambda = \frac{\sqrt{\nu - \nu_0(m)}}{C_0} \tag{28}$$

We have

$$v(x,y,t) = \sum_{m=0}^{\infty}\left[\int_0^{\infty} B_m(\Lambda)\cos\left(\sqrt{C_0^2\Lambda^2 + \nu_0(m)}\,t\right)\sin(\Lambda\,x)\,2C_0^2\Lambda d\Lambda\right]\cos\left(\frac{m\pi}{b}y\right) \tag{29}$$

At $t = 0$, we have

$$G(x,y) - \cos\left(\frac{\pi}{b}y\right) = \sum_{m=0}^{\infty}\left[\int_0^{\infty} B_m(\Lambda)2C_0^2\Lambda\sin(\Lambda\,x)\,d\Lambda\right]\cos\left(\frac{m\pi}{b}y\right) \tag{30}$$

Let us choose for simplicity

$$G(x,y) = g(x)\cos\left(\frac{\pi}{b}y\right) + \cos\left(\frac{\pi}{b}y\right) \tag{31}$$

then

$$g(x)\cos\left(\frac{\pi}{b}y\right) = \sum_{m=0}^{\infty}\left[\int_0^{\infty} B_m(\Lambda)2C_0^2\Lambda\sin(\Lambda\,x)\,d\Lambda\right]\cos\left(\frac{m\pi}{b}y\right) \tag{32}$$

and therefore $m = 1$ and

$$g(x) = 2C_0^2\int_0^{\infty}\Lambda B_1(\Lambda)\sin(\Lambda x)\,d\Lambda \tag{33}$$

This means that $2C_0^2\Lambda B_1(\Lambda)$ is the Fourier sine transform of $g(x)$. Let us choose $g(x)$ as (the choice should be such that the Fourier sine transform of this, yield a convergent integral in (29))

$$g(x) = \frac{x}{x^2+\alpha^2} \tag{34}$$

then

$$B_1(\Lambda) = \frac{e^{-\Lambda\alpha}}{2C_0^2\Lambda} \tag{35}$$

Now we substitute this into (29) to get

$$v(x,y,t) = \left[\int_0^{\infty} e^{-\Lambda\alpha}\cos\left(\sqrt{C_0^2\Lambda^2+\nu_0(1)}\,t\right)\sin(\Lambda\,x)\,d\Lambda\right]\cos\left(\frac{\pi}{b}y\right) \tag{36}$$

Combining this with the solution for $w(x,y,t)$ given in (13) we have

$$\eta(x,y,t) = \left[\cos\left(\sqrt{\nu_0(1)}\,t\right) + \int_0^{\infty} e^{-\Lambda\alpha}\cos\left(\sqrt{C_0^2\Lambda^2+\nu_0(1)}\,t\right)\sin(\Lambda\,x)\,d\Lambda\right]\cos\left(\frac{\pi}{b}y\right) \tag{37}$$

This solution assumes

$$\begin{aligned} G(x,y) &= \left(\frac{x}{x^2+\alpha^2}+1\right)\cos\left(\frac{\pi}{b}y\right) \\ F(y,t) &= \cos\left(\sqrt{\nu_0(1)}\,t\right)\cos\left(\frac{\pi}{b}y\right) \end{aligned} \tag{38}$$

3 The case that $F(y,t)$ is independent of t

In this case, one cannot use (6) because the solution (13) depends on time. Instead of (6), we should take

$$-C_0^2\left(\partial_x^2 w + \partial_y^2 w\right) + f^2 w = 0 \tag{39}$$

subject to

$$\begin{aligned}&w(0,y) = F(y),\\ &\partial_n w(x,0) = 0,\\ &\partial_n w(x,b) = 0,\\ &\lim_{x\to\infty} w(x,y) \text{ is bounded}\end{aligned} \tag{40}$$

The solution is given by

$$w(x,y) = \sum_{m=0}^{\infty} A_m e^{-(\nu_0(m)/C_0)x}\cos\left(\frac{m\pi}{b}y\right) \tag{41}$$

where $\nu_0(m)$ is given by (8) and

$$F(y) = \sum_{m=0}^{\infty} A_m \cos\left(\frac{m\pi}{b}y\right) \tag{42}$$

If we choose

$$F(y) = \cos\left(\frac{\pi}{b}y\right) \tag{43}$$

then

$$w(x,y) = e^{-(\nu_0(1)/C_0)x}\cos\left(\frac{\pi}{b}y\right) \tag{44}$$

This $w(x,y)$ will give the same PDE for v. The only condition affected is (16) which is now

$$v(x,y,0) = G(x) - e^{-(\nu_0(1)/C_0)x}\cos\left(\frac{\pi}{b}y\right) \tag{45}$$

For simplicity, we assume that

$$G(x) = g(x)\cos\left(\frac{\pi}{b}y\right) \tag{46}$$

and therefore

$$v(x,y,0) = \left(g(x) - e^{-(\nu_0(1)/C_0)x}\right)\cos\left(\frac{\pi}{b}y\right) \tag{47}$$

This condition is only used when we reach (30) where we now have

$$\left(g(x) - e^{-(\nu_0(1)/C_0)x} - 1\right)\cos\left(\frac{\pi}{b}y\right) = \sum_{m=0}^{\infty}\left[\int_0^{\infty} B_m(\Lambda) 2C_0^2\Lambda\sin(\Lambda\, x)\, d\Lambda\right]\cos\left(\frac{m\pi}{b}y\right) \tag{48}$$

Therefrore $m = 1$ and

$$g(x) - e^{-(\nu_0(1)/C_0)x} - 1 = \int_0^{\infty} B_1(\Lambda) 2C_0^2\Lambda\sin(\Lambda\, x)\, d\Lambda \tag{49}$$

If we now take

$$g(x) = \frac{x}{x^2+\alpha^2} + e^{-(\nu_0(1)/C_0)x} + 1 \tag{50}$$

then $B_1(\Lambda)$ is given by (35) and $v(x,y,t)$ is given by (36) as before. In this case we chose

$$\begin{aligned} G(x,y) &= \left(\frac{x}{x^2+\alpha^2} + e^{-(\nu_0(1)/C_0)x} + 1\right)\cos\left(\frac{\pi}{b}y\right) \\ F(y) &= \cos\left(\frac{\pi}{b}y\right) \end{aligned} \tag{51}$$

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